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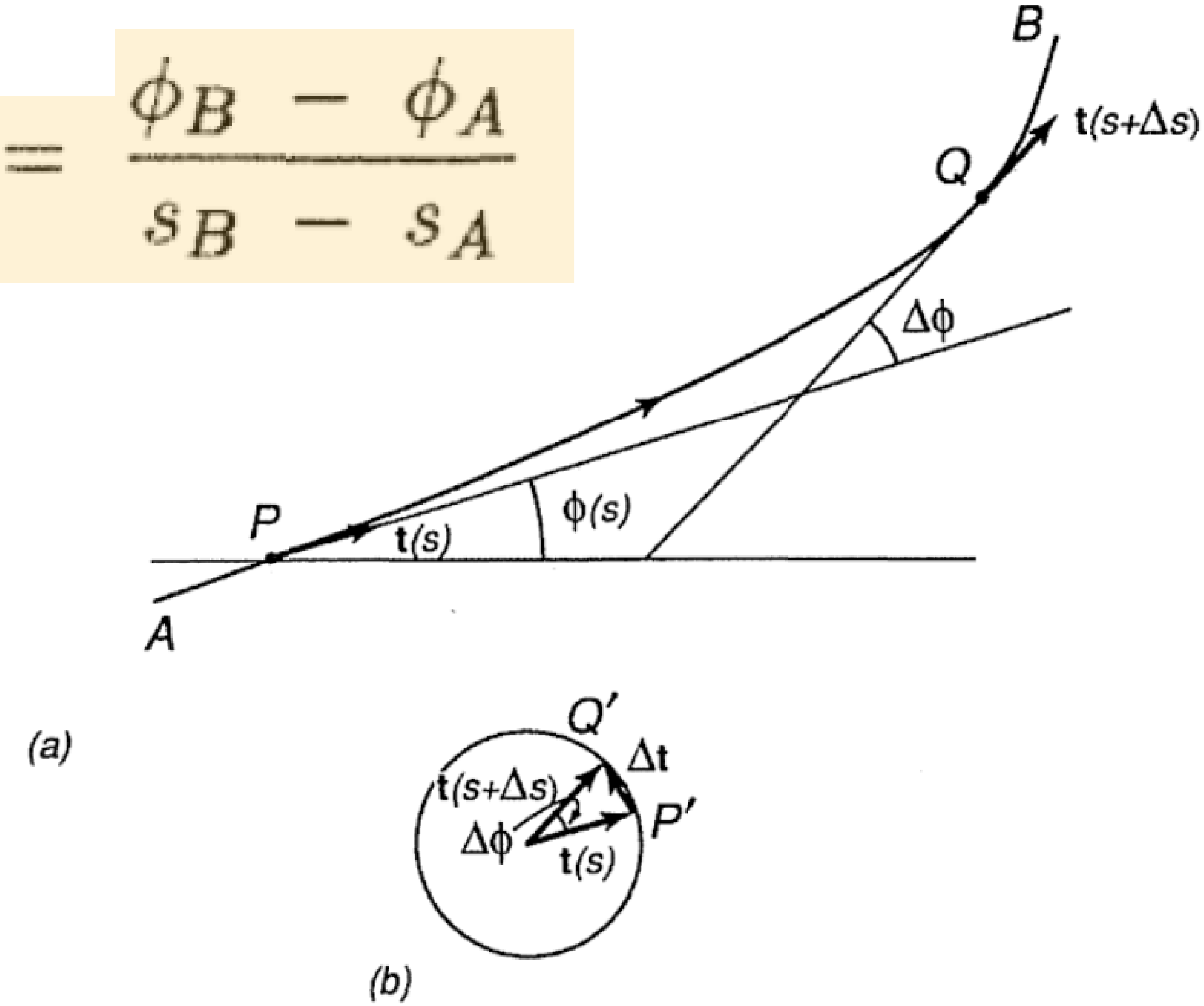
Exploring Curvature

With 141 Illustrations

$$\kappa_{avg} = \frac{\phi_B - \phi_A}{s_B - s_A}$$



it is the quotient of the length of the arc $P'Q'$ on the auxiliary circle and the length of the arc PQ on the curve whose curvature is being investigated.

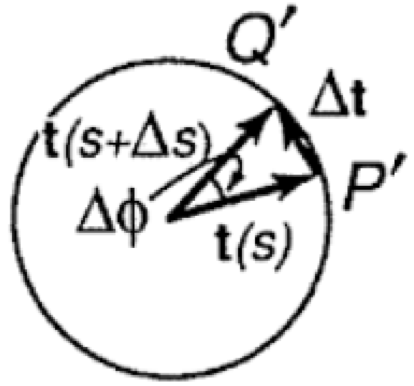
Remark 2 . The total curvature of an arc *AB* can be written in terms of the curvature κ as an integral, namely

$$\phi_B - \phi_A = \int_{s_A}^{s_B} \frac{d\phi}{ds}\, ds = \int_{s_A}^{s_B} \kappa\, ds \quad , \qquad (10.9)$$

where s_A and s_B are the values of the arc lengths at *A* and *B*. For this reason, the total curvature is also referred to as the *integral curvature* of an arc.

$$\frac{d\mathbf{t}}{ds} = \lim_{\Delta s \to 0} \frac{\mathbf{t}(s + \Delta s) - \mathbf{t}(s)}{\Delta s} . \qquad (10.10)$$

The quantity $\mathbf{t}(s + \Delta s) - \mathbf{t}(s)$ is the change in the unit tangent vector as one goes from P to Q and is represented by the vector $\Delta\mathbf{t}$ joining P' to Q' on the auxiliary circle in Fig. 68b. The derivative $d\mathbf{t}/ds$ is therefore a measure of the rate at which the unit tangent vector changes its direction as one moves along the curve.



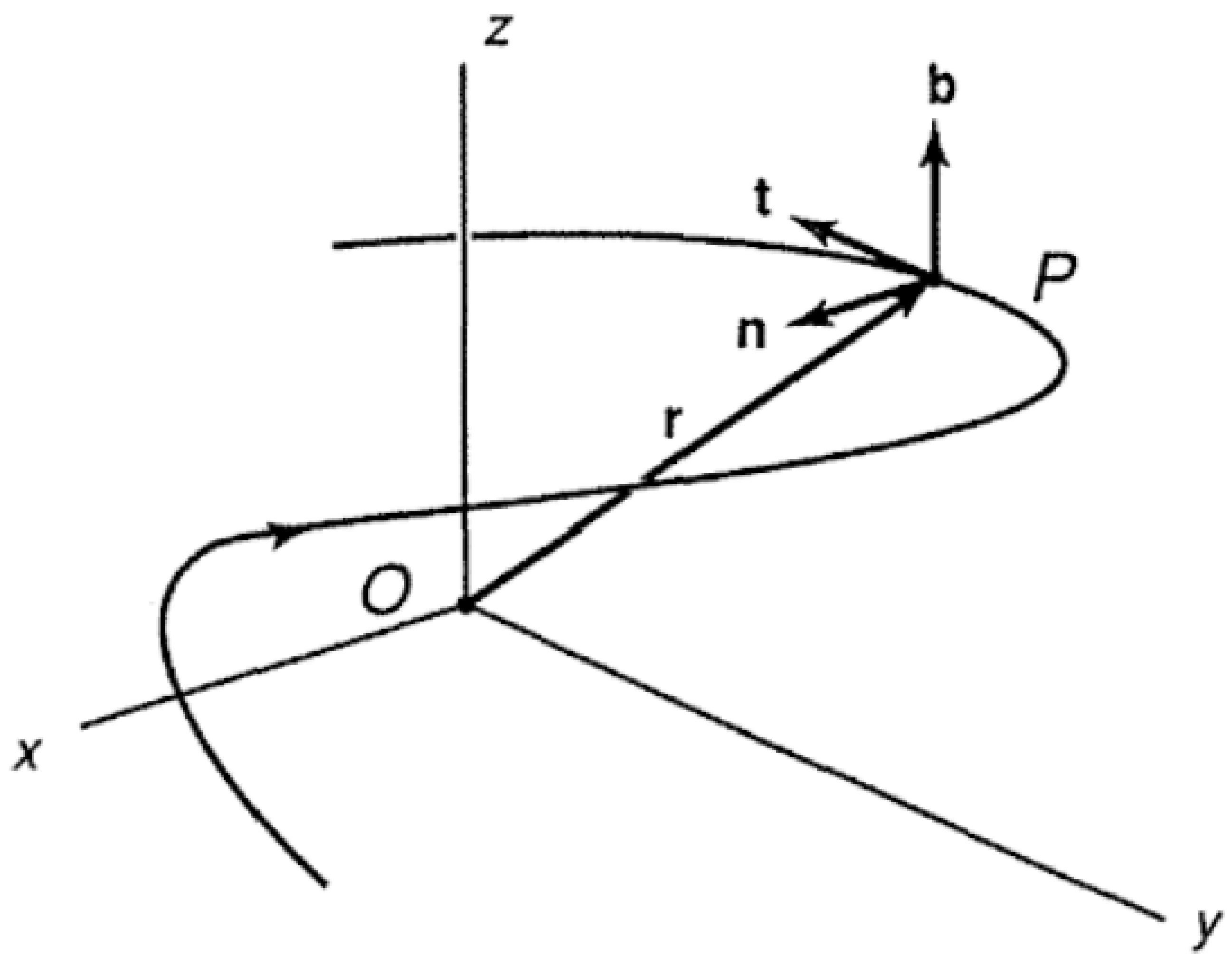
(b)

$$\frac{d\mathbf{t}}{ds} = \lim_{\Delta s \to 0} \left(\frac{\Delta \mathbf{t}}{\Delta \phi} \frac{\Delta \phi}{\Delta s} \right), \quad (10.11)$$

and making use of Equation (10.3), we will arrive at the expression

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} \ . \quad (10.12)$$

We may now think of $\mathbf{n}$ as a unit vector which is perpendicular, or *normal*, to the tangent to the curve AB at P in Fig. 68a. It is called the *principal unit normal vector*. The vector $d\mathbf{t}/ds$ is called the *curvature vector* (and is often denoted by a separate symbol). Equation (10.12) has



unit normal vector **n**. The plane of **t** and **n**, called the *osculating plane*, is no longer a fixed plane, but instead, rotates as we move along the curve. Let **b** be a unit vector perpendicular to the osculating plane, drawn as shown in Fig. 80. This vector is called the *unit binormal vector*. So, at each point along the curve, we now have a triad of unit vectors, **t**, **n**, **b**, making right-angles with one another.

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \quad ,$$

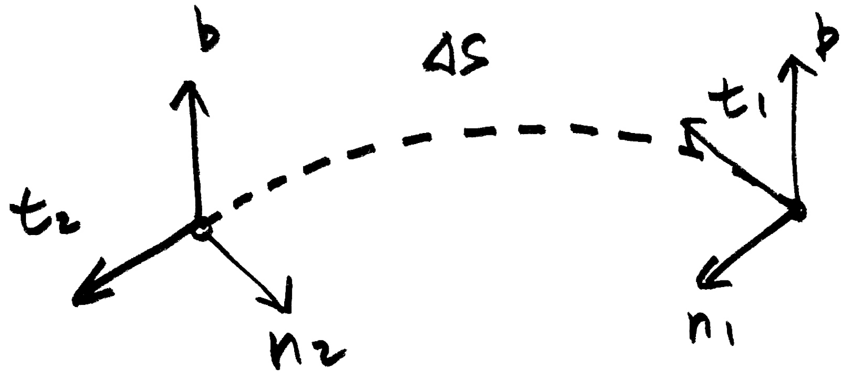
$$\frac{d\mathbf{n}}{ds} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b}\,, \qquad (10.13)$$

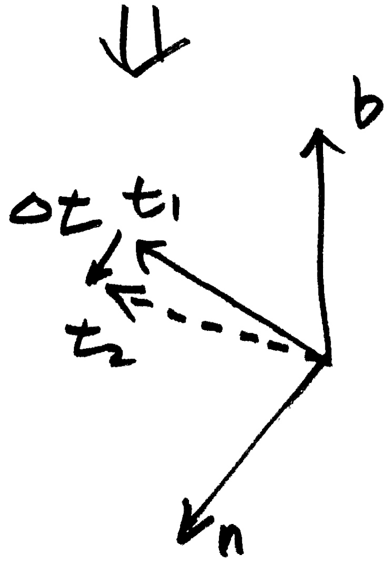
$$\frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n} \quad .$$

The first of these we have met before as Equation (10.12): it states that the rate at which $\mathbf{t}$ is changing with respect to arc length s is equal to the curvature κ times the principal unit normal $\mathbf{n}$. The last formula of the three shows that the rate at which the binormal $\mathbf{b}$ is changing can be expressed as a real number τ, called the *torsion*, times the unit vector $-\mathbf{n}$. Since $\mathbf{b}$ is perpendicular to the osculating plane, this equation implies that for $\tau \neq 0$ the normal to the osculating plane turns in the direction opposite to the unit vector $\mathbf{n}$. For a plane curve, both $d\mathbf{b}/ds$ and the torsion vanish. The second formula in the set describes how the principal normal is changing as a function of arc length: it has two components—one points opposite to $\mathbf{t}$, and the other points along $\mathbf{n}$. The Serret-Frenet formulae are the main tool for the analytical study of

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \quad ,$$

WHEN B IS CONSTANT (NOT CHANGING) I.E.
IN PLANAR CURVE.





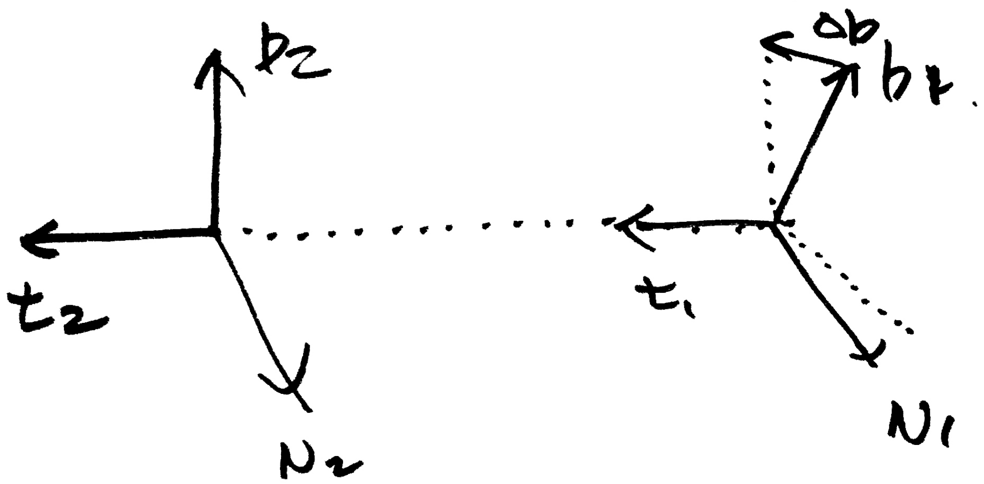
Δt BECOMES
PARRALLEL TO n.

$$\frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n}\ .$$

WHEN T IS CONSTANT / NOT CHAINGING

(IS NOT POSSIBLE THEORATICALLY THOUGH.)

→ IMAGINE TWISTING WIRE ?



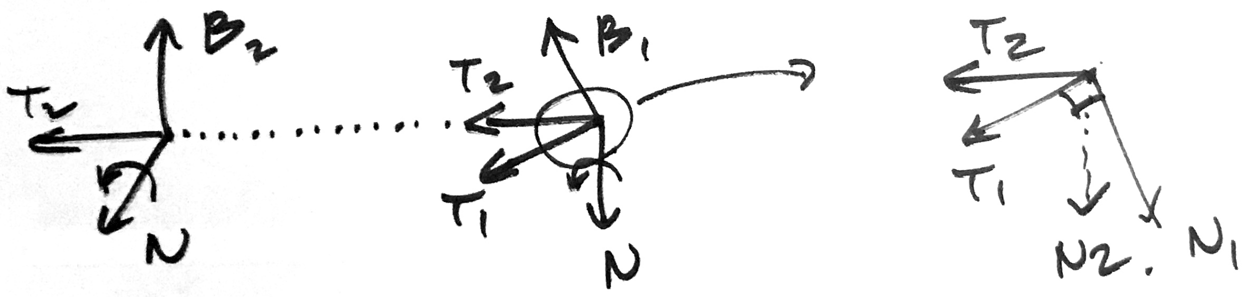
→ Δb BECOMES PARALLEL TO N.

IF N IS CONSTANT / NOT CHANGING

→ TWO OTHER VECTORS ROTATE ALONG N VECTOR
AS THE POINT ON A CURVE TRAVELS ALONG THE CRV

→ T VECTOR WILL CHANGES → THEN 'N' IS NOT CONSTANT !?

→ OR, 'N' CHANGES IT'S POSITION.



$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \quad ,$$

$$\frac{d\mathbf{n}}{ds} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b} \,, \qquad (10.13)$$

$$\frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n} \;.$$